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On meet and join matrices associated with incidence functions

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Abstract

We study recently meet matrices on meet-semilattices as an abstract generalization of greatest common divisor (GCD) matrices. Analogously, in this paper we consider join matrices on lattices as an abstract generalization of least common multiple (LCM) matrices. A formula for the determinant of join matrices on join-closed sets, bounds for the determinant of join matrices on all sets and a formula for the inverse of join matrices on join-closed sets are given. The concept of a semi-multiplicative function gives us formulae for meet matrices on join-closed sets and join matrices on meet-closed sets. Finally, we show what new the study of meet and join matrices contributes to the usual GCD and LCM matrices.

Keywords: GCD matrix; LCM matrix; Smith's determinant; Meet-semilattice; Meet matrix; Join matrix; Incidence function; Bounds for determinant; Inverse matrix; Semi-multiplicative function

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n positive integers with $x_1 < x_2 < \cdots < x_n$ and let f be an arithmetical function. Let (x_i, x_j) be the greatest common divisor (GCD) of x_i and x_j and define $(S)_f$ as the $n \times n$ matrix whose i, j entry is $f((x_i, x_j))$. We refer to $(S)_f$ as the GCD matrix on S with respect to f. The set S is said to be factor-closed if it contains every divisor of x for any $x \in S$. The set S

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is said to be gcd-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$. It is clear that a factor-closed set is always gcd-closed but not conversely. Let $[x_i, x_j]$ be the least common multiple (LCM) of x_i and x_j. The LCM matrix $[S]_f$ on S with respect to f and lcm-closed set are defined analogously.

Smith calculated $\det(S)_f$ on factor-closed sets [13, (5.)] and $\det[S]_f$ in a more special case [13, (3.)]. There is a large number of generalizations and analogues of these determinants in the literature. For a general account, see [8].

Bourque and Ligh [4] calculated $\det(S)_f$ and $(S)_f^{-1}$ on gcd-closed sets. They [5] also calculated $\det[S]_f$ and $[S]_f^{-1}$ on factor-closed sets. Haukkanen and Sillanpää [7] calculated $\det(S)_f$ and $\det[S]_f$ on gcd- and lcm-closed sets. Hong [9] provided a lower bound and an upper bound for $\det(S)_f$ whenever $f \in C_S$. He also provided a lower bound and an upper bound for $\det[S]_f$ whenever $(1/f) \in C_S$ and $f(x) \neq 0$ for all $x \in S$. Here $C_S = \{f : (x \in S, d \mid x) \Rightarrow (f * \mu)(d) > 0\}$, where μ is the number-theoretic Möbius function and $*$ is the Dirichlet convolution of arithmetical functions.

Haukkanen [6] studied meet matrices $(S)_f$ on S with respect to f as a lattice-theoretic generalization of GCD matrices. In [6] S is a subset of a meet-semilattice $P = (P, \leqslant)$ and f is a complex-valued function on P. This study offered, for example, a formula for $\det(S)_f$ on meet-closed set and a formula for $(S)_f^{-1}$ on lower-closed set. The concepts meet-closed and lower-closed are generalizations of gcd-closed and factor-closed respectively.

By using the Gram–Schmidt orthogonalization process as in [9], Korkee and Haukkanen [10] gave a lower bound and an upper bound for the determinant $\det(S)_f$ of meet matrices on meet-semilattices whenever $f \in C_S$. In this context $C_S = \{f : (x \in S, z \leqslant x) \Rightarrow (f * \mu)(\min P, z) > 0\}$, where μ is the Möbius function of P and $*$ is the convolution of incidence functions, see [11, p. 294–296]. They also provided a formula for $(S)_f^{-1}$ on meet-closed set whenever $f \in C_S$. In Section 3 we briefly review the recently achieved results concerning meet matrices.

In Section 4 we consider the dual of the lattice P and interpret $[S]_f$ as the dual of $(S)_f$. We provide the dual formulae for all the formulae given in Section 3. That is, we offer a formula for $\det[S]_f$ on join-closed set, a lower bound and an upper bound for $\det[S]_f$ whenever $f \in D_S$, and a formula for $[S]_f^{-1}$ on join-closed set whenever $f \in D_S$. The concepts of join-closed and D_S are the dual concepts of meet-closed and C_S respectively.

In Section 5 we assume that f is a semi-multiplicative function on P. We derive this concept from the theory of arithmetical functions, see [12]. This study gives us formulae similar to those presented in Sections 3 and 4 but in more complicated cases. We present bounds for the determinant of $(S)_f$ and formulae for the inverse of $(S)_f$ in join-closed sets and also bounds for the determinant of $[S]_f$ and formulae for the inverse of $[S]_f$ in meet-closed sets.

Finally, in Section 6 we revert to the usual number-theoretic setting and give many corollaries of the general case. Since GCD matrices have been examined literally, many of our results have already been found in this setting. The features of LCM matrices are less known, since—as we later note—the convolution of arithmetical functions is not always available. This section introduces many new results concerning LCM matrices. One new and remarkable result is a formula for $\det[S]_f$ on an lcm-closed set S without any restrictions on f. Further, for example, a formula for $[S]_f^{-1}$ on an lcm-closed set S is given.

2. Definitions

Let $(P, \leqslant) = (P, \wedge, \vee)$ be a finite lattice and let $S = \{x_1, x_2, \dots, x_n\}$ be a subset of P. We say that S is lower-closed if for every $x, y \in P$ with $x \in S$ and $y \leqslant x$ we have $y \in S$. We say that S is meet-closed if for every $x, y \in S$ we have $x \wedge y \in S$. We define the dual concepts upper-closed and join-closed analogously. It is clear that a lower-closed set is always meet-closed but not conversely, and dually, an upper-closed set is always join-closed but not conversely.

The order ideal generated by S is $\downarrow S = \{z \in P \mid \exists x \in S : z \leqslant x\}$. We define the dual order ideal generated by S as $\uparrow S = \{z \in P \mid \exists x \in S : x \leqslant z\}$. Obviously $\downarrow S$ is the minimal lower-closed set containing S and $\uparrow S$ is the minimal upper-closed set containing S.

Definition 2.1. Let f be a complex-valued function on P. Then the $n \times n$ matrix $(S)_f = (s_{ij})$, where

$$s_{ij} = f(x_i \wedge x_j), \tag{2.1}$$

is called the meet matrix on S with respect to f. Similarly, the $n \times n$ matrix $[S]_f = (s_{ij})$, where

$$s_{ij} = f(x_i \vee x_j), \tag{2.2}$$

is called the join matrix on S with respect to f.

Let f be a complex-valued function on $P \times P$ such that $f(x, y) = 0$ whenever $x \nleqslant y$. Then we say that f is an incidence function of P. If f and g are incidence functions of P, their sum $f + g$ is defined by $(f + g)(x, y) = f(x, y) + g(x, y)$ and their convolution $f * g$ is defined by $(f * g)(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z)g(z, y)$. The set of all incidence functions of P under addition and convolution forms a ring with unity, where the unity δ is defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. The incidence function ζ is defined by $\zeta(x, y) = 1$ if $x \leqslant y$, and $\zeta(x, y) = 0$ otherwise. The Möbius function μ of P is the inverse of ζ.

In what follows, let $(P, \leqslant)$ be a finite lattice and let $P = \{z_1, z_2, \dots, z_m\}$ with $z_i < z_j \Rightarrow i < j$. Let S be a subset of P and denote $S = \{x_1, x_2, \dots, x_n\} = \{z_{p_1}, z_{p_2}, \dots, z_{p_n}\}$ with $x_i = z_{p_i}$ and $x_i < x_j \Rightarrow i < j$.

3. Known results for meet matrices

Let f be a complex-valued function on P. In this section we associate f with a “restricted” incidence function f_d of $(P, \leqslant)$ by the formula $f(z) = f_\mathrm{d}(0, z)$, where $0 = z_1 = \min P$ and d means “down”. The function f_d can be used in the convolution of usual incidence functions when the first argument is equal to 0 and f_d is the left member in the convolution. Note that in [10] we used the same association but with a slightly different notation. In Sections 4 and 5 we use a completely different association. Also note that $(P, \leqslant)$ in [10] need not be a finite lattice but a meet-semilattice with finite principal order ideals. The present assumption that $(P, \leqslant)$ is a finite lattice is justified, because we will write the following propositions for dual lattice $(P', \preceq)$.

Definition 3.1 [10, Definition 2.1]. Let C_S denote the class of restricted incidence functions of $(P, \leqslant)$ defined as

$$C_S = \{f_\mathrm{d} \mid z \in \downarrow S \Rightarrow (f_\mathrm{d} * \mu)(0, z) > 0\}. \tag{3.1}$$

Proposition 3.1 [10, Lemma 3.2]. *Let $\downarrow S = \{w_1, w_2, \ldots, w_r\}$ with $w_i < w_j \Rightarrow i < j$ and let A denote the $n \times r$ matrix defined by*

$$a_{ij} = \begin{cases} \sqrt{(f_\mathrm{d} * \mu)(0, w_j)} & \text{if } w_j \leqslant x_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Then $(S)_f = AA^\mathrm{T}$.

Proposition 3.2 [10, Formula (4.1)]. *If S is meet-closed, then*

$$\det(S)_f = \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t < k}} (f_\mathrm{d} * \mu)(0, z). \tag{3.3}$$

Proposition 3.3 [6, Corollary 2, p. 116]. *If S is lower-closed, then*

$$\det(S)_f = \prod_{k=1}^{n} (f_\mathrm{d} * \mu)(0, x_k). \tag{3.4}$$

Proposition 3.4 [10, Theorem 5.1]. *If $f_\mathrm{d} \in C_S$, then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t < k}} (f_\mathrm{d} * \mu)(0, z) \tag{3.5}$$

and the equality holds if and only if S is meet-closed.

Proposition 3.5 [10, Lemma 6.1]. *If $f_\mathrm{d} \in C_S$, then $(S)_f$ is positive definite.*

Proposition 3.6 [10, Theorem 6.1]. *If* $f_{\mathrm{d}} \in C_S$, *then* $\det(S)_f \leqslant f(x_1)f(x_2)\cdots f(x_n)$.

Proposition 3.7 [10, Theorem 6.2]. *If* $f_{\mathrm{d}} \in C_S$, *then*

$$\det(S)_f \leqslant \frac{r!}{2}\left(1 - \frac{f(x_{a_1} \wedge \cdots \wedge x_{a_r})^r}{f(x_{a_1})\cdots f(x_{a_r})}\right)\prod_{k=1}^{n} f(x_k), \tag{3.6}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ *and* $2 \leqslant r \leqslant n$.

Proposition 3.8 [10, Theorem 7.1]. *Let S be a meet-closed set and let* $f_{\mathrm{d}} \in C_S$. *Then* $(S)_f$ *is invertible and*

$$\left((S)_f^{-1}\right)_{ij} = \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{1}{\sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (f_{\mathrm{d}} * \mu)(0, z)} \mu_S(x_i, x_k)\mu_S(x_j, x_k), \tag{3.7}$$

where $\mu_S = \zeta_S^{-1}$ *and* ζ_S *is the restriction of* ζ *on* $S \times S$.

Proposition 3.9 [6, Theorem 6]. *Let S be a lower-closed set and let* $f_{\mathrm{d}} \in C_S$. *Then* $(S)_f$ *is invertible and*

$$\left((S)_f^{-1}\right)_{ij} = \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{1}{(f_{\mathrm{d}} * \mu)(0, x_k)} \mu(x_i, x_k)\mu(x_j, x_k). \tag{3.8}$$

4. Join matrices

4.1. Basic notations

Rename the elements of P as $y_i = z_{m-i+1}$ and let $P' = \{y_1, y_2, \ldots, y_m\}$. Then $(P', \preceq) = (P', \curlywedge, \curlyvee)$ is the dual lattice of $(P, \leqslant)$ with $x \leqslant y \Leftrightarrow y \preceq x$. It is obvious that $y_i \prec y_j \Rightarrow i < j$.

We write ζ and μ for the zeta function and the Möbius function of $(P, \leqslant)$ and we write ζ' and μ' for the zeta function and the Möbius function of $(P', \preceq)$.

In what follows, let f always be a complex-valued function on P. We associate f with a "restricted" incidence function f_{u} on $(P, \leqslant)$ by the formula $f(z) = f_{\mathrm{u}}(z, 1)$ where $1 = z_m = \max P$ and u means "up". The function f_{u} can be used in the convolution of usual incidence functions when the second argument is equal to 1 and f_{u} is the right member in the convolution.

In the same way, we associate f with "restricted" incidence functions f'_{d} and f'_{u} on $(P', \preceq)$ by the formula $f(y) = f'_{\mathrm{d}}(0', y) = f'_{\mathrm{u}}(y, 1')$. Note that since $0 = 1'$ and $1 = 0'$, we have $f_{\mathrm{d}}(0, z_i) = f'_{\mathrm{u}}(z_i, 0)$ and $f_{\mathrm{u}}(z_i, 1) = f'_{\mathrm{d}}(1, z_i)$ for all $z_i \in P$.

4.2. Structure theorem for join matrix

Lemma 4.1. *Let g be an incidence function of P. Then*

$$g(x, y) = \sum_{x \leqslant z \leqslant y} (\mu * g)(z, y) \tag{4.1}$$

for all $x, y \in P$.

Lemma 4.1 is a direct consequence of the formula $g = \delta * g = (\zeta * \mu) * g = \zeta * (\mu * g)$. The following Lemma 4.2 is our dual result of Proposition 3.1.

Lemma 4.2. *Let $\uparrow S = \{w_1, w_2, \ldots, w_r\}$ with $w_i < w_j \Rightarrow i < j$ and let A denote the $n \times r$ matrix defined by*

$$a_{ij} = \begin{cases} \sqrt{(\mu * f_{\mathrm{u}})(w_j, 1)} & \text{if } x_i \leqslant w_j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Then $[S]_f = AA^{\mathrm{T}}$.

Proof. For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$ we have

$$\begin{aligned}(AA^{\mathrm{T}})_{ij} &= \sum_{k=1}^{r} a_{ik}a_{jk} = \sum_{\substack{x_i \leqslant w_k \leqslant 1 \\ x_j \leqslant w_k \leqslant 1}} (\mu * f_{\mathrm{u}})(w_k, 1) \\ &= \sum_{x_i \vee x_j \leqslant w_k \leqslant 1} (\mu * f_{\mathrm{u}})(w_k, 1).\end{aligned}$$

Now it follows from Lemma 4.1 that $(AA^{\mathrm{T}})_{ij} = f_{\mathrm{u}}(x_i \vee x_j, 1) = f(x_i \vee x_j)$. This completes the proof. □

4.3. On Möbius and zeta functions on the dual lattice

Lemma 4.3. *We have $\zeta(z_i, z_j) = \zeta'(z_j, z_i)$ and $\mu(z_i, z_j) = \mu'(z_j, z_i)$ for all z_i, $z_j \in P$.*

The proof of Lemma 4.3 is a simple application of induction if we use the recursive property of μ from [1, p. 141]. We leave the details of the proof to the reader.

Lemma 4.4. *We have $(\mu * f_{\mathrm{u}})(z_i, 1) = (f'_{\mathrm{d}} * \mu')(1, z_i)$ for all $z_i \in P$.*

Proof. By the formula $f(z_j) = f_{\mathrm{u}}(z_j, 1) = f'_{\mathrm{d}}(1, z_j)$ and Lemma 4.3 we have

$$(\mu * f_{\mathrm{u}})(z_i, 1) = \sum_{z_i \leqslant z_j \leqslant 1} \mu(z_i, z_j) f_{\mathrm{u}}(z_j, 1)$$
$$= \sum_{1 \preceq z_j \preceq z_i} f'_{\mathrm{d}}(1, z_j)\mu'(z_j, z_i) = (f'_{\mathrm{d}} * \mu')(1, z_i).$$

This completes the proof. □

4.4. Determinant of join matrices

Now we introduce a formula for $[S]_f$, where S is a join-closed subset of P. This is the dual result for Proposition 3.2.

Theorem 4.1. *If S is join-closed, then*

$$\det[S]_f = \prod_{k=1}^{n} \sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k<t}} (\mu * f_{\mathrm{u}})(z, 1). \tag{4.3}$$

Proof. Let $S = \{x_1, x_2, \ldots, x_n\} = \{z_{p_1}, z_{p_2}, \ldots, z_{p_n}\}$ be a join-closed set (more precisely $\vee$-closed). Then $S' = \{y_{m-p_n+1}, \ldots, y_{m-p_1+1}\}$. Let E denote the $n \times n$ matrix defined by

$$E = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{4.4}$$

Multiplying by E changes the rows or columns of the matrices in reverse order. Thus we have

$$(E(S')_f^{\mathrm{T}}E)_{ij} = ((S')_f)_{(n-j+1,n-i+1)}$$
$$= f(y_{m-p_j+1} \curlywedge y_{m-p_i+1}) = f(z_{p_i} \vee z_{p_j}) = ([S]_f)_{ij}. \tag{4.5}$$

Therefore $[S]_f = E(S')_f^{\mathrm{T}}E$ and $\det[S]_f = \det(S')_f$. Because S' is meet-closed (more precisely $\curlywedge$-closed), by Proposition 3.2 and Lemma 4.3 we have

$$\det[S]_f = \det(S')_f = \prod_{k=1}^{n} \sum_{\substack{y_j \preceq y_{m-p_k+1} \\ y_j \npreceq y_t \\ t<m-p_k+1}} (f'_{\mathrm{d}} * \mu')(0', y_j)$$
$$= \prod_{k=1}^{n} \sum_{\substack{z_{p_k} \leqslant z_{m-j+1} \\ z_{m-t+1} \nleqslant z_{m-j+1} \\ p_k<m-t+1}} (\mu * f_{\mathrm{u}})(z_{m-j+1}, 1).$$

If we write $m - t + 1 = s$ and $z_{m-j+1} = z$ and recall that $z_{p_k} = x_k$, then we have (4.3). This completes the proof. □

We obtain a much simpler formula if S is upper-closed. Thus we have the dual result of Proposition 3.3. For the proof we need the following lemma.

Lemma 4.5. *Let S be upper-closed and let g be a complex-valued function on P. Then*

$$g(x_k) = \sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k < t}} g(z) \tag{4.6}$$

for all $x_k \in S$.

Proof. Let $x_k \in S$, $x_k \leqslant z$ and $x_{k+1}, x_{k+2}, \ldots, x_n \nleqslant z$. Since S is upper-closed and $x_k \leqslant z$, we have $z = x_p$ for some $k \leqslant p \leqslant n$. Since $x_{k+1}, x_{k+2}, \ldots, x_n \nleqslant z$, we have $z = x_k$, which completes the proof. □

Corollary 4.1. *If S is upper-closed, then*

$$\det[S]_f = \prod_{k=1}^{n} (\mu * f_{\mathrm{u}})(x_k, 1). \tag{4.7}$$

Proof. If S is upper-closed, we can replace g with $\mu * f_{\mathrm{u}}$ in Lemma 4.5. Then Theorem 4.1 implies Corollary 4.1. This completes the proof. □

4.5. *Lower bound for* $\det[S]_f$

We shall present a dual result of Proposition 3.4. First we define and examine the set D_S, the dual concept of the set C_S.

Definition 4.1. Denote

$$D_S = \{f_{\mathrm{u}} \mid z \in \uparrow S \Rightarrow (\mu * f_{\mathrm{u}})(z, 1) > 0\}, \tag{4.8}$$

where the functions f_{u} are restricted incidence functions of $(P, \leqslant)$.

Example. If $g_{\mathrm{u}} = \zeta * f_{\mathrm{u}}$ and $f(x) > 0$ for all $x \in P$, then clearly $g_{\mathrm{u}} \in D_S$ for all S. As a special case, $\zeta^2 = \zeta * \zeta \in D_S$, where $\zeta^2(x, y)$ is the number of elements in the interval $[x, y]$. Also, $\zeta^k \in D_S$, $k \geqslant 2$, where $\zeta^k(x, y)$ is the number of x, y chains with repetitions of length k, see e.g. [1, p. 142].

Lemma 4.6. *Let $f_{\mathrm{u}} \in D_S$. Then $0 < f(y) < f(x)$ for all $x, y \in \uparrow S$ with $x < y$.*

Proof. Let $f_{\mathrm{u}} \in D_S$ and $x, y \in \uparrow S$ with $x < y$. Then $(\mu * f_{\mathrm{u}})(z, 1) > 0$ for all z such that $x \leqslant z$ and $y \leqslant z$. Thus by Lemma 4.1

$$\begin{aligned} 0 &< \sum_{y \leqslant z \leqslant 1} (\mu * f_{\mathrm{u}})(z, 1) = f_{\mathrm{u}}(y, 1) = f(y) \\ &< \sum_{x \leqslant z \leqslant 1} (\mu * f_{\mathrm{u}})(z, 1) = f_{\mathrm{u}}(x, 1) = f(x). \end{aligned}$$

This completes the proof. □

In Lemma 4.7 we will use the notation $C_{S'}$. To avoid misunderstandings we give its definition explicitly. Note that $\downarrow S' = \uparrow S$.

Definition 4.2. Denote

$$C_{S'} = \{f'_{\mathrm{d}} \mid z \in \downarrow S' \Rightarrow (f'_{\mathrm{d}} * \mu')(0', z) > 0\}, \tag{4.9}$$

where the functions f'_{d} are restricted incidence functions of $(P', \preceq)$.

Lemma 4.7. *If $f_{\mathrm{u}} \in D_S$, then $f'_{\mathrm{d}} \in C_{S'}$.*

Proof. Let $f_{\mathrm{u}} \in D_S$ and let $y_i \in S'$ and $z \preceq y_i$. Since $y_i \in S$ and $y_i \leqslant z$, we have $(\mu * f_{\mathrm{u}})(z, 1) > 0$. The proof is completed by noting that $(f'_{\mathrm{d}} * \mu')(1, z) = (\mu * f_{\mathrm{u}})(z, 1) > 0$ by Lemma 4.4, so $f'_{\mathrm{d}} \in C_{S'}$. □

We are now ready to give the dual result of Proposition 3.4.

Theorem 4.2. *If $f_{\mathrm{u}} \in D_S$, then*

$$\det[S]_f \geqslant \prod_{k=1}^{n} \sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k < t}} (\mu * f_{\mathrm{u}})(z, 1) \tag{4.10}$$

and the equality holds if and only if S is join-closed.

Proof. Let $f_{\mathrm{u}} \in D_S$. By Lemma 4.7 we have $f'_{\mathrm{d}} \in C_{S'}$. As in the proof of Theorem 4.1, we have $\det[S]_f = \det(S')_f$. Thus by Proposition 3.4

$$\det[S]_f = \det(S')_f \geqslant \prod_{k=1}^{n} \sum_{\substack{y_j \preceq y_{m-p_k+1} \\ y_j \npreceq y_t \\ t < m-p_k+1}} (f'_{\mathrm{d}} * \mu')(0', y_j)$$

and the equality holds if and only if $S' = \{y_{m-p_n+1}, \ldots, y_{m-p_2+1}, y_{m-p_1+1}\}$ is $\curlywedge$-closed. Again, as in the proof of Theorem 4.1, this is equivalent to

$$\det[S]_f = \det(S')_f \geqslant \prod_{k=1}^{n} \sum_{\substack{z_{p_k} \leqslant z \\ z_s \nleqslant z \\ p_k < s}} (\mu * f_{\rm u})(z, 1)$$

and the equality holds if and only if $S = \{z_{p_1}, z_{p_2}, \ldots, z_{p_n}\}$ is $\vee$-closed. This completes the proof. $\square$

4.6. Upper bound for $\det[S]_f$

Lemma 4.8. *If $f_{\rm u} \in D_S$, then $[S]_f$ is positive definite.*

Proof. Let $f_{\rm u} \in D_S$. Then $(\mu * f_{\rm u})(z, 1) > 0$ for all $z \in \uparrow S$. Define $S_i = \{x_1, x_2, \ldots, x_i\}$, $i = 1, 2, \ldots, n$. Then $f_{\rm u} \in D_{S_i}$ and by Theorem 4.2 we have

$$\det[S_i]_f \geqslant \prod_{k=1}^{i} \sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k < t}} (\mu * f_{\rm u})(z, 1) > 0,$$

where $i = 1, 2, \ldots, n$. Thus the principal minors of $[S]_f$ are positive. This completes the proof. $\square$

Lemma 4.8 is the dual result of Proposition 3.5. Now we give an upper bound for $\det[S]_f$, which is the dual result of Proposition 3.6.

Theorem 4.3. *If $f_{\rm u} \in D_S$, then* $\det[S]_f \leqslant f(x_1)f(x_2)\cdots f(x_n)$.

Now we give a new upper bound for $\det[S]_f$, which is the dual result of Proposition 3.7. We omit the proof, since it is similar to the proof of Proposition 3.7, see the proof of Theorem 6.2 in [10].

Theorem 4.4. *If $f_{\rm u} \in D_S$, then*

$$\det[S]_f \leqslant \frac{r!}{2}\left(1 - \frac{f(x_{a_1} \vee \cdots \vee x_{a_r})^r}{f(x_{a_1})\cdots f(x_{a_r})}\right)\prod_{k=1}^{n} f(x_k), \tag{4.11}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ *and* $2 \leqslant r \leqslant n$.

Note that the new upper bound (4.11) is sharper than what we find in Theorem 4.3 if we choose $r = 2$. Actually, $x_{a_1} \leqslant x_{a_1} \vee x_{a_2}$ and $x_{a_2} \leqslant x_{a_1} \vee x_{a_2}$, where the equalities cannot hold simultaneously. So, by Lemma 4.6 we have

$$0 < \frac{f(x_{a_1} \vee x_{a_2})^2}{f(x_{a_1})f(x_{a_2})} < 1.$$

4.7. Inverse of $[S]_f$

Now we introduce a formula for $[S]_f^{-1}$ when S is join-closed, which is the dual result of Proposition 3.8. We denote the restriction of ζ on $S \times S$ by ζ_S and let $\mu_S = \zeta_S^{-1}$. In the same way, we denote the restriction of ζ' on $S' \times S'$ by $\zeta'_{S'}$ and let $\mu'_{S'} = (\zeta'_{S'})^{-1}$. Note that by Lemma 4.3 we have $\mu_S(x_i, x_j) = \mu'_{S'}(x_j, x_i)$ for all $x_i, x_j \in S$.

Theorem 4.5. *If $f_\mathrm{u} \in D_S$, then $[S]_f$ is invertible. Furthermore,*

$$\left([S]_f^{-1}\right)_{ij} = \sum_{x_k \leqslant x_i \wedge x_j} \frac{1}{\sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k<t}} (\mu * f_\mathrm{u})(z, 1)} \mu_S(x_k, x_i)\mu_S(x_k, x_j) \tag{4.12}$$

if S is join-closed.

Proof. Let $S = \{z_{p_1}, z_{p_2}, \ldots, z_{p_n}\}$ be $\vee$-closed and let $f_\mathrm{u} \in D_S$. Then $S' = \{y_{m-p_n+1}, \ldots, y_{m-p_1+1}\}$ is $\curlywedge$-closed and we have $f'_\mathrm{d} \in C_{S'}$ by Lemma 4.7. By formula (4.5) we have $[S]_f = E(S')_f^\mathrm{T} E$, where E is the matrix defined in (4.4). Therefore, by Proposition 3.8 and Lemma 4.3 we have

$$\begin{aligned}
\left([S]_f^{-1}\right)_{ij} &= \left((S')_f^{-1}\right)_{(n-j+1,n-i+1)} \\
&= \sum_{\substack{y_{m-p_i+1} \preceq y_{m-p_k+1} \\ y_{m-p_j+1} \preceq y_{m-p_k+1}}} \frac{\mu_{S'}(y_{m-p_i+1}, y_{m-p_k+1})\mu_{S'}(y_{m-p_j+1}, y_{m-p_k+1})}{\sum_{\substack{z \preceq y_{m-p_k+1} \\ z \npreceq y_t \\ t<m-p_k+1}} (f'_\mathrm{d} * \mu')(0', z)} \\
&= \sum_{\substack{z_{p_k} \leqslant z_{p_i} \\ z_{p_k} \leqslant z_{p_j}}} \frac{1}{\sum_{\substack{z_{p_k} \leqslant z \\ z_t \nleqslant z \\ p_k<t}} (\mu * f_\mathrm{u})(z, 1)} \mu_S(z_{p_k}, z_{p_i})\mu_S(z_{p_k}, z_{p_j}).
\end{aligned}$$

This completes the proof. $\square$

We will also give a formula for $[S]_f^{-1}$ when S is upper-closed. For this purpose we need the following lemma. The proof is simple to derive by induction. We omit the details of the proof.

Lemma 4.9. *Let S be an upper-closed subset of P and let g and h be incidence functions of P. Let g_S and h_S be the restrictions of g and h on $S \times S$. Then $g_S + h_S$ and $g_S * h_S$ are the restrictions of $g + h$ and $g * h$ on $S \times S$. Furthermore, if g is invertible, then g_S is invertible and g_S^{-1} is the restriction of g^{-1} on $S \times S$.*

Let S be upper-closed in Theorem 4.5. Now by Lemma 4.9 we can replace expressions $\mu_S(x_i, x_j)$ with $\mu(x_i, x_j)$. By Lemma 4.5 we have

$$\sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k<t}} (\mu * f_\mathrm{u})(z, 1) = (\mu * f_\mathrm{u})(x_k, 1)$$

for all $x_k \in S$. Now we can present the dual result of Proposition 3.9.

Corollary 4.2. *If $f_{\mathrm{u}} \in D_S$, then $[S]_f$ is invertible. Furthermore,*

$$\left([S]_f^{-1}\right)_{ij} = \sum_{x_k \leqslant x_i \wedge x_j} \frac{1}{(\mu * f_{\mathrm{u}})(x_k, 1)} \mu(x_k, x_i)\mu(x_k, x_j) \tag{4.13}$$

if S is upper-closed.

Note that if each incidence function is associated with an appropriate matrix, then the sum and the convolution of incidence functions become the ordinary matrix sum and product, see [1, p. 139]. This means that formulae (4.12) and (4.13) can be easily written as products of three matrices.

5. Meet and join matrices with respect to semi-multiplicative functions

5.1. Definitions

In this section we examine join matrices on meet-closed sets and meet matrices on join-closed sets. We assume that the associated function f on P is semi-multiplicative. The concept of a semi-multiplicative function on P is a generalization of the known concept of a semi-multiplicative arithmetical function [12, p. 49].

Definition 5.1. Let f be a complex-valued function on P. We say that f is a semi-multiplicative function if

$$f(x)f(y) = f(x \wedge y)f(x \vee y) \tag{5.1}$$

for all $x, y \in P$.

Remark. In this section let f always be a semi-multiplicative function on P such that $f(x) \neq 0$ for all $x \in P$.

The complex-valued function $1/f$ on P is defined by $(1/f)(x) = 1/f(x)$. If g is an incidence function of P, then the incidence function $1/g$ of P is defined similarly. One can easily show that f is semi-multiplicative if and only if $1/f$ is semi-multiplicative.

The assumption that f is semi-multiplicative and that $f(x) \neq 0$ for all $x \in P$ gives us a way to express a join matrix in terms of a certain meet matrix and a meet matrix in terms of a certain join matrix, see Sections 5.2 and 5.4. Namely, now $f(x \wedge y)$ can be written in terms of $f(x)$, $f(y)$ and $f(x \vee y)$, and vice versa, since they are all nonzero.

5.2. Join matrix in terms of a certain meet matrix

Lemma 5.1. *Let* $D = \operatorname{diag}(f(x_1), \ldots, f(x_n))$. *Then* $[S]_f = D(S)_{1/f}D$.

Proof. Since

$$(D(S)_{1/f}D)_{ij} = f(x_i)((S)_{1/f})_{ij}f(x_j) = \frac{f(x_i)f(x_j)}{f(x_i \wedge x_j)} = f(x_i \vee x_j),$$

we have $[S]_f = D(S)_{1/f}D$. This completes the proof. □

We can now convert all the results on meet matrices presented in Section 3 to join matrices. The basic idea is found in [9, p. 318].

5.3. On join matrices with respect to semi-multiplicative functions

The next two results are easily derived by using Propositions 3.2 and 3.3. We do not present the latter proof, since it is similar to the first.

Theorem 5.1. *If S is meet-closed, then*

$$\det[S]_f = \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} \left(\left(\frac{1}{f}\right)_{\mathrm{d}} * \mu\right)(0, z). \tag{5.2}$$

Proof. Since S is meet-closed, we have by Lemma 5.1 and Proposition 3.2 that

$$\begin{aligned}\det[S]_f &= (\det D)^2 \det(S)_{1/f} = \left(\prod_{k=1}^{n} f(x_k)^2\right)\left(\prod_{k=1}^{n} \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} \left(\left(\frac{1}{f}\right)_{\mathrm{d}} * \mu\right)(0, z)\right) \\ &= \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} \left(\left(\frac{1}{f}\right)_{\mathrm{d}} * \mu\right)(0, z).\end{aligned}$$

This completes the proof. □

Corollary 5.1. *If S is lower-closed, then*

$$\det[S]_f = \prod_{k=1}^{n} f(x_k)^2 \left(\left(\frac{1}{f}\right)_{\mathrm{d}} * \mu\right)(0, x_k). \tag{5.3}$$

The next result is derived by using Proposition 3.4 but we omit the proof, since it is similar to the proof of Theorem 5.1. Note that we have two assumptions $f(x) \neq 0$ for all $x \in P$ and $(1/f)_{\mathrm{d}} \in C_S$, and neither is strong enough to imply the other.

Theorem 5.2. *Let* $(1/f)_{\mathrm{d}} \in C_S$. *Then*

$$\det[S]_f \geqslant \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} \left(\left(\frac{1}{f} \right)_{\mathrm{d}} * \mu \right)(0, z) \tag{5.4}$$

and the equality holds if and only if S is meet-closed.

We present the proof of the next theorem, although it is not so tricky either.

Theorem 5.3. *Let* $(1/f)_{\mathrm{d}} \in C_S$. *Then*

$$\det[S]_f \leqslant \frac{r!}{2} \left(1 - \frac{f(x_{a_1}) \cdots f(x_{a_r})}{f(x_{a_1} \wedge \cdots \wedge x_{a_r})^r} \right) \prod_{k=1}^{n} f(x_k) \tag{5.5}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ *and* $2 \leqslant r \leqslant n$.

Proof. Since $(1/f)_{\mathrm{d}} \in C_S$, we have by Lemma 5.1 and Proposition 3.7 that

$$\begin{aligned} \det[S]_f &= (\det D)^2 \det(S)_{1/f} \\ &\leqslant \left(\prod_{k=1}^{n} f(x_k)^2 \right) \frac{r!}{2} \left(1 - \frac{\frac{1}{f}(x_{a_1} \wedge \cdots \wedge x_{a_r})^r}{\frac{1}{f}(x_{a_1}) \cdots \frac{1}{f}(x_{a_r})} \right) \prod_{k=1}^{n} \frac{1}{f}(x_k) \\ &= \frac{r!}{2} \left(1 - \frac{f(x_{a_1}) \cdots f(x_{a_r})}{f(x_{a_1} \wedge \cdots \wedge x_{a_r})^r} \right) \prod_{k=1}^{n} f(x_k), \end{aligned}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ and $2 \leqslant r \leqslant n$. This completes the proof. □

We next present a formula for $[S]_f^{-1}$ on a meet-closed set.

Theorem 5.4. *Let* $(1/f)_{\mathrm{d}} \in C_S$. *Then* $[S]_f$ *is invertible. Furthermore,*

$$\left([S]_f^{-1} \right)_{ij} = \frac{1}{f(x_i) f(x_j)} \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{\mu_S(x_i, x_k) \mu_S(x_j, x_k)}{\sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (((1/f))_{\mathrm{d}} * \mu)(0, z)} \tag{5.6}$$

if S is meet-closed.

Proof. Since $[S]_f = D(S)_{1/f} D$, where $D = \operatorname{diag}(f(x_1), \ldots, f(x_n))$, and $(1/f)_{\mathrm{d}} \in C_S$, the inverse of $[S]_f$ exists by Proposition 3.4. Furthermore, since S is meet-closed, we have by Proposition 3.8 that

$$\begin{aligned} \left([S]_f^{-1} \right)_{ij} &= (D^{-1})_{ii} \left((S)_{1/f}^{-1} \right)_{ij} \left(D^{-1} \right)_{jj} \\ &= \frac{1}{f(x_i) f(x_j)} \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{\mu_S(x_i, x_k) \mu_S(x_j, x_k)}{\sum_{\substack{z \leqslant x_k \\ z \nleqslant x_t \\ t<k}} (((1/f))_{\mathrm{d}} * \mu)(0, z)}. \end{aligned}$$

This completes the proof. □

Finally, we present a formula for $[S]_f^{-1}$ on lower-closed set. The proof is easy to derive by using Lemma 5.1 and Proposition 3.9.

Corollary 5.2. *Let* $(1/f)_{\mathrm{d}} \in C_S$. *Then* $[S]_f$ *is invertible. Furthermore,*

$$\left([S]_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{\substack{x_i \leqslant x_k \\ x_j \leqslant x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{(((1/f))_{\mathrm{d}} * \mu)(0, x_k)} \tag{5.7}$$

if S is lower-closed.

5.4. Meet matrix in terms of a certain join matrix

Lemma 5.2. *Let* $D = \mathrm{diag}(f(x_1), \ldots, f(x_n))$. *Then* $(S)_f = D[S]_{1/f}D$.

Proof. Since

$$\left(D[S]_{1/f}D\right)_{ij} = f(x_i)\left([S]_{1/f}\right)_{ij} f(x_j) = \frac{f(x_i)f(x_j)}{f(x_i \vee x_j)} = f(x_i \wedge x_j),$$

we have $(S)_f = D[S]_{1/f}D$. This completes the proof. □

We can now convert all the results on join matrices found in Section 4 to meet matrices.

5.5. On meet matrices with respect to semi-multiplicative functions

The next three results are easily derived by using Theorem 4.1, Corollary 4.1 and Theorem 4.2 respectively. We omit the proofs, since they are all similar to the proof of Theorem 5.1.

Theorem 5.5. *If S is join-closed, then*

$$\det(S)_f = \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k<t}} \left(\mu * \left(\frac{1}{f}\right)_{\mathrm{u}}\right)(z, 1). \tag{5.8}$$

Corollary 5.3. *If S is upper-closed, then*

$$\det(S)_f = \prod_{k=1}^{n} f(x_k)^2 \left(\mu * \left(\frac{1}{f}\right)_{\mathrm{u}}\right)(x_k, 1). \tag{5.9}$$

Theorem 5.6. *Let* $(1/f)_{\mathrm{u}} \in D_S$. *Then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k<t}} \left(\mu * \left(\frac{1}{f}\right)_{\mathrm{u}}\right)(z,1) \tag{5.10}$$

and the equality holds if and only if S is join-closed.

The next theorem is easy to derive by using Theorem 4.4. We omit the proof, since it is similar to the proof of Theorem 5.3.

Theorem 5.7. *Let* $(1/f)_{\mathrm{u}} \in D_S$. *Then*

$$\det(S)_f \leqslant \frac{r}{2}\left(1 - \frac{f(x_{a_1})\cdots f(x_{a_r})}{f(x_{a_1} \vee \cdots \vee x_{a_r})^r}\right) \prod_{k=1}^{n} f(x_k), \tag{5.11}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ *and* $2 \leqslant r \leqslant n$.

We next present two formulae for $(S)_f^{-1}$ on join-closed set and on upper-closed set. They are easily derived by using Theorem 4.5 and Corollary 4.2 respectively. The proofs are similar to the proof of Theorem 5.4.

Theorem 5.8. *Let* $(1/f)_{\mathrm{u}} \in D_S$. *Then* $(S)_f$ *is invertible. Furthermore*,

$$\left((S)_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \leqslant x_i \wedge x_j} \frac{\mu_S(x_k,x_i)\mu_S(x_k,x_j)}{\sum_{\substack{x_k \leqslant z \\ x_t \nleqslant z \\ k<t}} \left(\mu * \left(\frac{1}{f}\right)_{\mathrm{u}}\right)(z,1)} \tag{5.12}$$

if S is join-closed.

Corollary 5.4. *If* $(1/f)_{\mathrm{u}} \in D_S$, *then* $(S)_f$ *is invertible. Furthermore*,

$$\left((S)_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k \leqslant x_i \wedge x_j} \frac{\mu(x_k,x_i)\mu(x_k,x_j)}{\left(\mu * \left(\frac{1}{f}\right)_{\mathrm{u}}\right)(x_k,1)} \tag{5.13}$$

if S is upper-closed.

6. Results for GCD and LCM matrices

6.1. Definitions for GCD matrices

In this section we present our results in the language of number theory. It is well known that the set of positive integers $\mathbf{Z}_+$ with the usual divisibility relation $|$ is an infinite lattice with finite principal order ideals. In the lattice $(\mathbf{Z}_+, |)$ we have $x \wedge y = (x, y)$ and $x \vee y = [x, y]$. Therefore, in Section 3, the notations used in the lattice $(P, \leqslant)$ can directly be replaced with those used in $(\mathbf{Z}_+, |)$, so it is a simple task to convert the results for meet matrices to GCD matrices.

Let $(P, \leqslant) = (\mathbf{Z}_+, |)$ and let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n positive integers with $x_1 < x_2 < \cdots < x_n$. The notations in Section 3 are now converted as follows. First, $x \wedge y = (x, y)$ and $x_1 \wedge x_2 \wedge \cdots \wedge x_n = (x_1, x_2, \ldots, x_n)$ means the GCD of the elements $x_1, x_2, \ldots, x_n$ of S. Further, $\downarrow x = \{y \in P : y \mid x\}$ is the set of all divisors of x and $\downarrow S = \{y \in P : (\exists x \in S : y \mid x)\}$. The concepts of meet-closed and lower-closed sets can be replaced with the concepts of gcd-closed and factor-closed sets and the concept of a meet matrix can be replaced with the concept of a GCD matrix, see Section 1.

In the number theory complex-valued functions f on $\mathbf{Z}_+$ are called arithmetical functions. In this section we denote the usual number-theoretic Möbius function by μ and the Möbius incidence function of P by μ_P. It is well known that $\mu_P(x, y) = \mu(y/x)$ for all $x, y \in P$ with $x \mid y$, see [11, p. 300]. Thus

$$\begin{aligned}(f_{\mathrm{d}} * \mu_P)(0, z) &= \sum_{0 \leqslant y \leqslant z} f_{\mathrm{d}}(0, y)\mu_P(y, z) = \sum_{1|y|z} f_{\mathrm{d}}(1, y)\mu_P(y, z) \\ &= \sum_{1|y|z} f(y)\mu(z/y) = (f * \mu)(z),\end{aligned} \tag{6.1}$$

where the binary operation $*$ on the left side of (6.1) means the convolution of incidence functions and the binary operation $*$ on the right side stands for the Dirichlet convolution of arithmetical functions. The definition of C_S now has the form

$$C_S = \{f : (x \in S, d \mid x) \Rightarrow (f * \mu)(d) > 0\}. \tag{6.2}$$

We saw that, for example, $N^a \in C_S$, where $a \geqslant 1$ and $N^a(n) = n^a$ for all n. An arithmetical function f is called completely multiplicative if $f(x)f(y) = f(xy)$ for all $x, y \in P$, see [2, p. 33]. Obviously function N^a is completely multiplicative. We shall refer to this function later.

The next lemma shows that the incidence function term $\mu_S(x_i, x_j)$ used in our inverse formulae corresponds the term c_{ij} given in [4, Theorem 3].

Lemma 6.1. *Let S be gcd-closed and let*

$$c_{ij} = \begin{cases} \sum_{\substack{dx_i | x_j \\ dx_i \nmid x_t \\ t<j}} \mu(d) & \text{if } x_i \mid x_j, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Then $\mu_S(x_i, x_j) = c_{ij}$ for all $x_i, x_j \in S$.

Proof. The case $x_i \nmid x_j$ is trivial. Let $x_i \mid x_j$ and define the incidence function F of S by $F(x_i, x_j) = c_{ij}$. Now

$$(F * \zeta_S)(x_i, x_j) = \sum_{x_i | x_r | x_j} F(x_i, x_r) = \sum_{x_i | x_r | x_j} \sum_{\substack{dx_i | x_r \\ dx_i \nmid x_t \\ t<r}} \mu(d).$$

Using the method adopted in the proof of [6, p. 114] we can say that

$$\sum_{x_i | x_r | x_j} \sum_{\substack{dx_i | x_r \\ dx_i \nmid x_t \\ t<r}} \mu(d) = \sum_{dx_i | x_j} \mu(d).$$

By an elementary property of μ we have

$$\sum_{dx_i | x_j} \mu(d) = \sum_{d \left| \frac{x_j}{x_i} \right.} \mu(d) = \begin{cases} 1 & \text{if } x_j = x_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $F * \zeta_S = \delta_S$ and further $F(x_i, x_j) = \mu_S(x_i, x_j) = c_{ij}$ for all $x_i, x_j \in S$. This completes the proof. □

After this we are ready to convert the results of Section 3 to the number-theoretic setting. However, before this we take a look at the notions relating with LCM matrices.

6.2. *Definitions for LCM matrices*

In order to utilize the results of join matrices to $(\mathbf{Z}_+, |)$, we use the following idea. First, we assume that S has already been chosen and let $s = [x_1, x_2, \ldots, x_n]$ denote the LCM of the elements $x_1, x_2, \ldots, x_n$ of S. Second, let $T_s = \downarrow s$ be the set of all divisors of s. Since T_s is finite, lower- and upper-closed, we can use the lattice $(T_s, |)$ instead of the lattice $(\mathbf{Z}_+, |)$. In the language of lattice theory, $(T_s, |)$ is a sublattice of $(\mathbf{Z}_+, |)$, see [3, p. 7]. It does not matter that the examined lattice $(T_s, |)$ actually depends on S. We possess the finite lattice $(T_s, |)$ and the finite subset S of T_s, so the results found for join matrices given in Section 4 are valid for $(T_s, |)$ and S. After explaining some details it is again easy to convert these results for join matrices to LCM matrices.

The results relating to semi-multiplicative functions of P given in Section 5 can also be converted if we note that the concept of a semi-multiplicative function is adapted from number theory.

Remark. In this section let $(P, \leqslant) = (T_s, |)$ always be the finite lattice, where $T_s = \downarrow s$ is the set of all divisors of s.

The notations in Section 4 are now converted as follows. First, $x \vee y = [x, y]$. Further, $\uparrow x = \{y \in P : x \mid y \mid s\}$ and $\uparrow S = \{y \in P : (\exists x \in S : \ x \mid y \mid s)\}$. The concept of join-closed set can be replaced by the concept of lcm-closed set. We say that S is multiple-closed if for every $x, y \in P$ with $x \in S$ and $x \mid y \mid s$ we have

$y \in S$. Note that the concept of multiple-closed set has not previously been given in the literature and it is formed only to obtain a special case of the concept of upper-closed set. Note also that $x_n = s$ if S is lcm- or multiple-closed. So one has $x \mid \max S$ for any $x \in S$ if S is lcm- or multiple-closed. The concept of join matrix reduces to the concept of LCM matrix. We recall that the $n \times n$ matrix $[S]_f = (s_{ij})$, where $s_{ij} = f([x_i, x_j])$, is called the LCM matrix on S with respect to f.

As a more specialized conversion we have that

$$(\mu_P * f_{\mathrm{u}})(z, 1) = \sum_{z \leqslant y \leqslant 1} \mu_P(z, y) f_{\mathrm{u}}(y, 1) = \sum_{z|y|s} f(y)\mu(y/z). \tag{6.4}$$

Note that the right side of (6.4) is no longer presented in terms of Dirichlet convolution. By (6.4) the definition of D_S has the form

$$D_S = \left\{ f : (x \in S, x \mid d \mid s) \Rightarrow \sum_{d|e|s} f(e)\mu(e/d) > 0 \right\}. \tag{6.5}$$

An interesting way to construct functions belonging to D_S is given below.

Lemma 6.2. *Let f be a completely multiplicative function. If $f \in C_P$, then $1/f \in D_P$.*

Proof. Let $f \in C_P$ be a completely multiplicative function. Then $(f * \mu)(x) > 0$, $f(x) = \sum_{d|x} (f * \mu)(d) > 0$ and further $1/f(x) > 0$ for all $x \in P$. Now

$$\begin{aligned} \sum_{d|e|s} \frac{1}{f}(e)\mu(e/d) &= \frac{1}{f}(s) \sum_{\frac{e}{d} | \frac{s}{d}} \mu(e/d) f\left(\frac{s}{d} \Big/ \frac{e}{d}\right) \\ &= \frac{1}{f}(s)(\mu * f)(s/d) = \frac{1}{f}(s)(f * \mu)(s/d) > 0 \end{aligned}$$

for all $x \in P$ with $x \mid d \mid s$. Therefore $(1/f) \in D_P$. This completes the proof. $\square$

Example. Since $J_a(x) = (N^a * \mu)(x) > 0$ for all $x \in \mathbf{Z}_+$, where J_a is the well-known Jordan totient, we have $N^a \in C_P$. Further, since N^a is completely multiplicative, we have $N^{-a} \in D_P$.

We also have a formula for μ_S on lcm-closed sets. It has almost the same structure as (6.3).

Lemma 6.3. *Let S be lcm-closed and let*

$$c_{ij} = \begin{cases} \sum_{\substack{dx_i | x_j \\ dx_t \nmid x_j \\ i < t}} \mu(d) & \text{if } x_i \mid x_j, \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

Then $\mu_S(x_i, x_j) = c_{ij}$ for all $x_i, x_j \in S$.

Proof. The case $x_i \nmid x_j$ is trivial. Let

$$U = \left\{ \frac{s}{x_n}, \ldots, \frac{s}{x_1} \right\}.$$

Since $\frac{s}{x_i} \mid s$ for all $x_i \in S$, the sets S and U belong to the same P. Let $x_i \mid x_j$. Then the intervals $\{d \in P : x_i \mid d \mid x_j\}$ and $\{d \in P : \frac{s}{x_j} \mid d \mid \frac{s}{x_i}\}$ are isomorphic and since U is gcd-closed, we have by Lemma 6.1 that

$$\begin{aligned} \mu_S(x_i, x_j) &= \mu_U\left(\frac{s}{x_j}, \frac{s}{x_i}\right) \\ &= \sum_{\substack{\frac{ds}{x_j} \mid \frac{s}{x_i} \\ \frac{ds}{x_j} \nmid \frac{s}{x_t} \\ i<t}} \mu(d) = \sum_{\substack{dx_i \mid x_j \\ dx_t \nmid x_j \\ i<t}} \mu(d) = c_{ij}. \end{aligned}$$

This completes the proof. □

6.3. New results for GCD matrices

In this section we present corollaries in the number-theoretic setting, which are completely new. The following corollary is a special case of Proposition 3.7 and it is easy to derive by using (6.1).

Corollary 6.1. *If $f \in C_S$, then*

$$\det(S)_f \leqslant \frac{r!}{2}\left(1 - \frac{f((x_{a_1}, \ldots, x_{a_r}))^r}{f(x_{a_1}) \cdots f(x_{a_r})}\right) \prod_{k=1}^{n} f(x_k), \tag{6.7}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ and $2 \leqslant r \leqslant n$.

In Section 5 it was assumed that f is a semi-multiplicative function. If we assume that $f(1) \neq 0$, then f is a so-called quasi-multiplicative function. (Actually, if we denote "is a special case of" by $<$, we have that completely multiplicativity $<$ multiplicativity $<$ quasi-multiplicativity $<$ semi-multiplicativity, see [12, p. 49], [2, p. 33] and [7, p. 239].) In Section 5 we also assumed that $f(x) \neq 0$ for all $x \in P$. Therefore we call f a quasi-multiplicative function rather than a semi-multiplicative function.

The following four corollaries are special cases of Theorem 5.5, Corollary 5.3, Theorem 5.6 and Theorem 5.7 respectively. These (determinant formulae) are easy to derive by using (6.4). Note that the first corollary is a generalization of Theorem 3.3 given by Haukkanen and Sillanpää in [7]. We present the details later (see Corollary 6.13).

Corollary 6.2. *Let S be lcm-closed and let f be a quasi-multiplicative function such that $f(x) \neq 0$ for all $x \in P$. Then*

$$\det(S)_f = \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k|z|x_n \\ x_t \nmid z \\ k<t}} \sum_{z|y|x_n} \frac{1}{f}(y)\mu(y/z). \tag{6.8}$$

Corollary 6.3. *Let S be multiple-closed and let f be a quasi-multiplicative function such that $f(x) \neq 0$ for all $x \in P$. Then*

$$\det(S)_f = \prod_{k=1}^{n} f(x_k)^2 \sum_{x_k|y|x_n} \frac{1}{f}(y)\mu(y/x_k). \tag{6.9}$$

Corollary 6.4. *Let f be a quasi-multiplicative function such that $1/f \in D_S$ and $f(x) \neq 0$ for all $x \in P$. Then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k|z|s \\ x_t \nmid z \\ k<t}} \sum_{z|y|s} \frac{1}{f}(y)\mu(y/z) \tag{6.10}$$

and the equality holds if and only if S is lcm-closed.

Corollary 6.5. *Let f be a quasi-multiplicative function such that $1/f \in D_S$ and $f(x) \neq 0$ for all $x \in P$. Then*

$$\det(S)_f \leqslant \frac{r!}{2}\left(1 - \frac{f(x_{a_1})\cdots f(x_{a_r})}{f([x_{a_1},\ldots,x_{a_r}])^r}\right)\prod_{k=1}^{n} f(x_k), \tag{6.11}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ and $2 \leqslant r \leqslant n$.

The next two corollaries are special cases of Theorem 5.8 and Corollary 5.4 respectively. These (inverse formulae) are easy to derive by using (6.4) and Lemma 6.3.

Corollary 6.6. *Let f be a quasi-multiplicative function such that $1/f \in D_S$ and $f(x) \neq 0$ for all $x \in P$. Then $(S)_f$ is invertible. Furthermore, if S is lcm-closed, then*

$$\left((S)_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k|(x_i,x_j)} \frac{c_{ki}c_{kj}}{\delta_k}, \tag{6.12}$$

where

$$\delta_k = \sum_{\substack{x_k|z|x_n \\ x_t \nmid z \\ k<t}} \sum_{z|y|x_n} \frac{1}{f}(y)\mu(y/z), \quad c_{kr} = \sum_{\substack{dx_k|x_r \\ dx_t \nmid x_r \\ k<t}} \mu(d). \tag{6.13}$$

Corollary 6.7. *Let f be a quasi-multiplicative function such that $1/f \in D_S$ and $f(x) \neq 0$ for all $x \in P$. Then $(S)_f$ is invertible. Furthermore, if S is multiple-closed, then*

$$\left((S)_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{x_k|(x_i,x_j)} \frac{\mu(x_i/x_k)\mu(x_j/x_k)}{\sum_{x_k|y|x_n}(1/f)(y)\mu(y/x_k)}. \tag{6.14}$$

6.4. Known results for GCD matrices

In this section we list corollaries which have previously been given in the literature. The general forms are given in Section 3. Propositions 3.2, 3.3 and 3.4 respectively are generalizations of the following three corollaries. These (determinant formulae) can be found directly by using (6.1). Note that Corollary 6.9 is the famous Smith's determinant formula.

Corollary 6.8 [4, Theorem 2]. *If S is gcd-closed, then*

$$\det(S)_f = \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_t \\ t<k}} (f * \mu)(d). \tag{6.15}$$

Corollary 6.9 [13, (5.)]. *If S is factor-closed, then*

$$\det(S)_f = \prod_{k=1}^{n} (f * \mu)(x_k). \tag{6.16}$$

Corollary 6.10 [9, Theorem 1]. *If $f \in C_S$, then*

$$\det(S)_f \geqslant \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_t \\ t<k}} (f * \mu)(d) \tag{6.17}$$

and the equality holds if and only if S is gcd-closed.

Propositions 3.8 and 3.9 are generalizations of the following two corollaries. They are easy to derive by using (6.1) and Lemma 6.1.

Corollary 6.11 [4, Theorem 3]. *If $f \in C_S$, then $(S)_f$ is invertible. Furthermore, if S is gcd-closed, then*

$$\left((S)_f^{-1}\right)_{ij} = \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{c_{ik}c_{jk}}{\delta_k}, \tag{6.18}$$

where

$$\delta_k = \sum_{\substack{d|x_k \\ d\nmid x_t \\ t<k}} (f * \mu)(d), \quad c_{rk} = \sum_{\substack{dx_r|x_k \\ dx_r\nmid x_t \\ t<k}} \mu(d). \tag{6.19}$$

Corollary 6.12 [4, Corollary 1]. *If $f \in C_S$, then $(S)_f$ is invertible. Furthermore, if S is factor-closed, then*

$$\left((S)_f^{-1}\right)_{ij} = \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{1}{(f * \mu)(x_k)} \mu(x_k/x_i)\mu(x_k/x_j). \tag{6.20}$$

Finally in this section we show that Corollary 6.2 is a generalization of the following corollary presented by Haukkanen and Sillanpää [7, Theorem 3.3].

Corollary 6.13. *Let S be lcm-closed and let f be a completely multiplicative function such that $f(x) \neq 0$ for all $x \in P$. Then*

$$\det(S)_f = f(x_n)^{-n} \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{d|\frac{x_n}{x_k} \\ d\nmid \frac{x_n}{x_t} \\ k<t}} (f * \mu)(d). \tag{6.21}$$

Proof. Since f is completely multiplicative, we have $f(x)f(y) = f(xy)$ for all $x, y \in P$. The function f is also quasi-multiplicative, so by Corollary 6.2 we have

$$\begin{aligned}
\det(S)_f &= \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k|z|x_n \\ x_t\nmid z \\ k<t}} \sum_{z|y|x_n} \frac{1}{f}(y)\mu(y/z) \\
&= \frac{1}{f(x_n)^n} \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k|z|x_n \\ x_t\nmid z \\ k<t}} \sum_{1|\frac{y}{z}|\frac{x_n}{z}} \mu(y/z) f(x_n/y) \\
&= \frac{1}{f(x_n)^n} \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{x_k|z|x_n \\ x_t\nmid z \\ k<t}} (\mu * f)(x_n/z) \\
&= \frac{1}{f(x_n)^n} \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{\frac{x_n}{z}|\frac{x_n}{x_k} \\ \frac{x_n}{z}\nmid \frac{x_n}{x_t} \\ k<t}} (f * \mu)(x_n/z).
\end{aligned}$$

This completes the proof. □

6.5. New results for LCM matrices

In this section we present results which are also completely new in the number-theoretic setting. First we convert results from Section 4. The following four corollaries are special cases of Theorem 4.1, Corollary 4.1, Theorem 4.2 and Theorem 4.4 respectively and they are easy to derive by using (6.4). Note that Corollary 6.14 is a generalization of Theorem 3.4 given by Haukkanen and Sillanpää in [7]. We explain this result later in detail (see Corollary 6.27). Note that now we calculate $\det[S]_f$ on an lcm-closed set for any arithmetical function f, not only for completely multiplicative functions f as done in [7].

Corollary 6.14. *If S is lcm-closed, then*

$$\det[S]_f = \prod_{k=1}^{n} \sum_{\substack{x_k|z|x_n \\ x_t\nmid z \\ k<t}} \sum_{z|y|x_n} f(y)\mu(y/z). \tag{6.22}$$

Corollary 6.15. *If S is multiple-closed, then*

$$\det[S]_f = \prod_{k=1}^{n} \sum_{x_k|y|x_n} f(y)\mu(y/x_k). \tag{6.23}$$

Corollary 6.16. *If $f \in D_S$, then*

$$\det[S]_f \geqslant \prod_{k=1}^{n} \sum_{\substack{x_k|z|s \\ x_t\nmid z \\ k<t}} \sum_{z|y|s} f(y)\mu(y/z) \tag{6.24}$$

and the equality holds if and only if S is lcm-closed.

Corollary 6.17. *If $f \in D_S$, then*

$$\det[S]_f \leqslant \frac{r!}{2}\left(1 - \frac{f([x_{a_1},\ldots,x_{a_r}])^r}{f(x_{a_1})\cdots f(x_{a_r})}\right)\prod_{k=1}^{n} f(x_k), \tag{6.25}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ and $2 \leqslant r \leqslant n$.

The next two corollaries are special cases of Theorem 4.5 and Corollary 4.2 respectively. These (inverse formulae) are easy to derive by using (6.4) and Lemma 6.3.

Corollary 6.18. *If $f \in D_S$, then $[S]_f$ is invertible. Furthermore, if S is lcm-closed, then*

$$\left([S]_f^{-1}\right)_{ij} = \sum_{x_k|(x_i,x_j)} \frac{c_{ki}c_{kj}}{\delta_k}, \tag{6.26}$$

where

$$\delta_k = \sum_{\substack{x_k|z|x_n \\ x_t\nmid z \\ k<t}} \sum_{z|y|x_n} f(y)\mu(y/z), \quad c_{kr} = \sum_{\substack{dx_k|x_r \\ dx_t\nmid x_r \\ k<t}} \mu(d). \tag{6.27}$$

Corollary 6.19. *If $f \in D_S$, then $[S]_f$ is invertible. Furthermore, if S is multiple-closed, then*

$$\left([S]_f^{-1}\right)_{ij} = \sum_{x_k|(x_i,x_j)} \frac{1}{\sum_{x_k|y|x_n} f(y)\mu(y/x_k)} \mu(x_i/x_k)\mu(x_j/x_k). \tag{6.28}$$

We now convert results from Section 5. The following three corollaries are special cases of Corollary 5.1, Theorem 5.3 and Theorem 5.4 respectively and they are easy to derive by using (6.1) and Lemma 6.1. Note the slight difference between the assumption on f in [5, Theorem 2] and in Corollary 6.20. We consider this further in the next section (see Corollary 6.24).

Corollary 6.20. *Let S be factor-closed and let f be a quasi-multiplicative function such that $f(x) \neq 0$ for all $x \in P$. Then*

$$\det[S]_f = \prod_{k=1}^{n} f(x_k)^2 \left(\frac{1}{f} * \mu\right)(x_k). \tag{6.29}$$

Corollary 6.21. *Let f be a quasi-multiplicative function such that $(1/f) \in C_S$ and $f(x) \neq 0$ for all $x \in P$. Then*

$$\det[S]_f \leqslant \frac{r!}{2}\left(1 - \frac{f(x_{a_1})\cdots f(x_{a_r})}{f((x_{a_1},\ldots,x_{a_r}))^r}\right)\prod_{k=1}^{n} f(x_k) \tag{6.30}$$

whenever $1 \leqslant a_1 < \cdots < a_r \leqslant n$ and $2 \leqslant r \leqslant n$.

Corollary 6.22. *Let f be a quasi-multiplicative function such that $(1/f) \in C_S$ and $f(x) \neq 0$ for all $x \in P$. Then $[S]_f$ is invertible. Furthermore, if S is gcd-closed, then*

$$\left([S]_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{c_{ik}c_{jk}}{\delta_k}, \tag{6.31}$$

where

$$\delta_k = \sum_{\substack{d|x_k \\ d\nmid x_t \\ t<k}} \left(\frac{1}{f} * \mu\right)(d), \quad c_{rk} = \sum_{\substack{dx_r|x_k \\ dx_r\nmid x_t \\ t<k}} \mu(d). \tag{6.32}$$

6.6. Known results for LCM matrices

In this section we list corollaries which have previously been given in the literature. The general forms are given in Section 5. These formulae can be found directly by using (6.1). Theorem 5.1 is a generalization of the following corollary.

Corollary 6.23 [7, Theorem 3.2]. *Let S be gcd-closed and let f be a quasi-multiplicative function such that $f(x) \neq 0$ for all $x \in P$. Then*

$$\det[S]_f = \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{z|x_k \\ z\nmid x_t \\ t<k}} \left(\frac{1}{f} * \mu\right)(z). \tag{6.33}$$

The next three corollaries are generalizations of Corollary 6.20 (and thus Corollary 5.1), Theorem 5.2 and Corollary 6.22 (and thus Corollary 5.2) respectively. Note that in Corollary 6.20 we assume that f is quasi-multiplicative and now that f is multiplicative. The assumption of multiplicativity is thus superfluous. However, Bourque and Ligh assume that f is multiplicative but actually use only quasi-multiplicativity, see the discussion above Theorem 2 in [5].

Corollary 6.24 [5, Theorem 2]. *Let S be factor-closed and let f be a multiplicative function such that $f(x) \neq 0$ for all $x \in P$. Then*

$$\det[S]_f = \prod_{k=1}^{n} f(x_k)^2 \left(\frac{1}{f} * \mu\right)(x_k). \tag{6.34}$$

Corollary 6.25 [9, Theorem 2]. *Let f be a quasi-multiplicative function such that $(1/f) \in C_S$ and $f(x) \neq 0$ for all $x \in P$. Then*

$$\det[S]_f \geqslant \prod_{k=1}^{n} f(x_k)^2 \sum_{\substack{z|x_k \\ z\nmid x_t \\ t<k}} \left(\frac{1}{f} * \mu\right)(z) \tag{6.35}$$

and the equality holds if and only if S is gcd-closed.

Corollary 6.26 [5, Theorem 2(ii)]. *Let f be a quasi-multiplicative function such that $(1/f) \in C_S$ and $f(x) \neq 0$ for all $x \in P$. Then $[S]_f$ is invertible. Furthermore, if S is factor-closed, then*

$$\left([S]_f^{-1}\right)_{ij} = \frac{1}{f(x_i)f(x_j)} \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{\mu(x_k/x_i)\mu(x_k/x_j)}{\left(\frac{1}{f} * \mu\right)(x_k)}. \tag{6.36}$$

Finally, in this section we note that Corollary 6.14 (and thus Theorem 4.1) are generalizations of the following corollary given by Haukkanen and Sillanpää in

[7, Theorem 3.4]. We leave the proof as an elementary exercise on arithmetical functions, since it is quite similar to the proof of Corollary 6.13.

Corollary 6.27 [7, Theorem 3.4]. *Let f be a completely multiplicative function such that $f(x) \neq 0$ for all x. If S is lcm-closed, then*

$$\det[S]_f = f(x_n)^n \prod_{k=1}^{n} \sum_{\substack{d \mid \frac{x_n}{x_k} \\ d \nmid \frac{x_n}{x_t} \\ k<t}} \left(\frac{1}{f} * \mu\right)(d). \tag{6.37}$$

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